

Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD

P.A. Boyle,¹ N.H. Christ,² N. Garron,³ E.J. Goode,⁴ T. Janowski,⁴
C. Lehner,⁵ Q. Liu,² A.T. Lytle,⁴ C.T. Sachrajda,⁴ A. Soni,⁶ and D. Zhang²

(The RBC and UKQCD Collaborations)

¹*SUPA, School of Physics, The University of Edinburgh, Edinburgh EH9 3JZ, UK*

²*Physics Department, Columbia University, New York, NY 10027, USA*

³*School of Mathematics, Trinity College, Dublin 2, Ireland*

⁴*School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, UK*

⁵*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA*

⁶*Brookhaven National Laboratory, Upton, NY 11973, USA*

There has been much speculation as to the origin of the $\Delta I = 1/2$ rule ($\text{Re}A_0/\text{Re}A_2 \simeq 22.5$). We find that the two dominant contributions to the $\Delta I = 3/2$, $K \rightarrow \pi\pi$ correlation functions have opposite signs leading to a significant cancellation. This partial cancellation occurs in our computation of $\text{Re}A_2$ with physical quark masses and kinematics (where we reproduce the experimental value of A_2) and also for heavier pions at threshold. For $\text{Re}A_0$, although we do not have results at physical kinematics, we do have results for pions at zero-momentum with $m_\pi \simeq 420$ MeV ($\text{Re}A_0/\text{Re}A_2 = 9.1(2.1)$) and $m_\pi \simeq 330$ MeV ($\text{Re}A_0/\text{Re}A_2 = 12.0(1.7)$). The contributions which partially cancel in $\text{Re}A_2$ are also the largest ones in $\text{Re}A_0$, but now they have the same sign and so enhance this amplitude. The emerging explanation of the $\Delta I = 1/2$ rule is a combination of the perturbative running to scales of $O(2 \text{ GeV})$, a relative suppression of $\text{Re}A_2$ through the cancellation of the two dominant contributions and the corresponding enhancement of $\text{Re}A_0$. QCD and EWP penguin operators make only very small contributions at such scales.

PACS numbers: 11.15.Ha, 11.30.Er 12.38.Gc 13.25.Es

Introduction

The “ $\Delta I = 1/2$ rule” remains one of the longest-standing puzzles in particle physics. It refers to the surprising feature that in $K \rightarrow \pi\pi$ decays the final state is about 450 times more likely to have total isospin $I=0$ than $I=2$. In terms of the (predominantly real) $K \rightarrow \pi\pi$ amplitudes A_0 and A_2 , where the suffix denotes I , this corresponds to $\text{Re}A_0/\text{Re}A_2 \simeq 22.5$. Perturbative running from the electroweak scale to about 1.5–2 GeV contributes a factor of approximately 2 to this ratio [1, 2]; the remaining factor of about 10 should come from non-perturbative QCD or, just possibly, from new physics. Lattice QCD provides the opportunity for the non-perturbative evaluation of A_0 and A_2 , although it is only very recently that such direct $K \rightarrow \pi\pi$ calculations have become feasible. In this letter we summarise the emerging explanation of the $\Delta I = 1/2$ rule from computations of A_0 and A_2 by the RBC-UKQCD collaboration.

The first results from direct simulations of a kaon decaying into two pions were presented in [3–5]. The determination of A_0 , where the two pions have vacuum quantum numbers, is particularly challenging and so far it has not been calculated with physical masses and momenta. We are striving to overcome technical issues such as the efficient evaluation of disconnected diagrams and the projection of the physical state through the use of G-parity boundary conditions [6–9] in order to evaluate A_0 at physical kinematics in the near future. In the meantime we have evaluated A_0 and A_2 for pions with masses of approximately 420 MeV [3] and 330 MeV [10] at thresh-

old, i.e. with the pions at rest. For these unphysical masses we do find a significant enhancement of the ratio $\text{Re}A_0/\text{Re}A_2$, albeit a smaller one than 22.5 (see the first two rows of Table I). While investigating the origin of this enhancement we found a surprising cancellation in the evaluation of $\text{Re}A_2$, which significantly increases the ratio $\text{Re}A_0/\text{Re}A_2$. This suppression of $\text{Re}A_2$ is the main result presented here.

We have also evaluated A_2 with physical masses and momenta, obtaining a result for $\text{Re}A_2$ which agrees with the physical value and determining $\text{Im}A_2$ for the first time [4, 5] (see the third row of Table 1). In the evaluation of $\text{Re}A_2$ at physical kinematics there is a similar cancellation; indeed it is even more pronounced than at the unphysical masses in the first two rows of Tab. I.

In the next section we summarize the simulations we have performed, highlighting features of immediate relevance for the $\Delta I = 1/2$ rule and referring to earlier publications for other details. We then explain the partial cancellation of the two contributions to $\text{Re}A_2$, which contradicts naïve expectations from the factorization (vacuum insertion) hypothesis. We also show that these two contributions have the same sign in $\text{Re}A_0$. We conclude by explaining how these features combine to provide an emerging understanding of the $\Delta I = 1/2$ rule. Of course a full quantitative explanation will require a calculation of $\text{Re}A_0$ at physical kinematics which is underway.

Calculation of the Decay Amplitudes

Our evidence is based on calculations from three Domain Wall Fermion (DWF) ensembles with 2+1 sea-quark

	a^{-1} [GeV]	m_π [MeV]	m_K [MeV]	$\text{Re}A_2$ [10^{-8} GeV]	$\text{Re}A_0$ [10^{-8} GeV]	$\frac{\text{Re}A_0}{\text{Re}A_2}$	notes
16^3 Iwasaki	1.73(3)	422(7)	878(15)	4.911(31)	45(10)	9.1(2.1)	threshold calculation
24^3 Iwasaki	1.73(3)	329(6)	662(11)	2.668(14)	32.1(4.6)	12.0(1.7)	threshold calculation
IDSDR	1.36(1)	142.9(1.1)	511.3(3.9)	1.38(5)(26)	-	-	physical kinematics
Experiment	-	135-140	494-498	1.479(4)	33.2(2)	22.45(6)	

TABLE I: Summary of simulation parameters and results obtained on three DWF ensembles.

flavours (see Tab.I). Papers [4, 5] describe a complete calculation of A_2 using the IDSDR (Iwasaki + Dislocation Suppressing Determinant Ratio) gauge action [11] for (almost) physical pion and kaon masses and realistic kinematics. The ensemble was generated at a single lattice spacing a ($a^{-1} \simeq 1.4$ GeV) chosen so that the volume is sufficiently large to accommodate the propagation of physical pions. In [3] a complete calculation of both A_0 and A_2 was carried out with the Iwasaki gauge action for $m_\pi \simeq 422$ MeV and $m_K \simeq 737, 878$ and 1117 MeV (here we present results for $m_K \simeq 878$ MeV which corresponds to almost energy-conserving decays). Although the calculation was performed at threshold, this was the first time a signal for $\text{Re}A_0$ had been obtained in the direct evaluation of the $K \rightarrow \pi\pi$ matrix elements. A similar threshold calculation was presented in [10] on a larger volume (24^3) with $m_\pi = 329$ MeV. The increased time extent of this lattice suppresses “around-the-world” effects in which one of the pions from the sink propagates in the forward time direction, crossing the periodic boundary and reaching the weak operator with the kaon. The calculation also used two-pion sources in which the single-pion wall sources are separated in time by a small number of time slices δ (the results presented here are for $\delta = 4$). We find that this suppresses the (unphysical) vacuum contributions in the $I = 0$ channel, significantly reducing the noise. In this way $\text{Re}A_0$ was resolved using only 138 configurations, compared to 800 in [3].

The amplitudes A_0 and A_2 can be expressed in terms of the “master formula”

$$A_I = F_I \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} \sum_{j=1}^7 \left[(z_i(\mu) + \tau y_i(\mu)) Z_{ij}^{\text{lat} \rightarrow \overline{\text{MS}}} M_j^{\Delta I, \text{lat}} \right] \quad (I = 0, 2). \quad (1)$$

$\tau = -V_{ts}^* V_{td} / V_{us}^* V_{ud}$ and the V_{ij} are elements of the Cabibbo-Kobayashi-Maskawa matrix. $M_i^{\Delta I, \text{lat}} \equiv \langle (\pi\pi)_I | Q_i^{\text{lat}} | K \rangle$ are the matrix elements calculated on the lattice. They are determined by fitting three-point correlation functions composed of a kaon source at $t = 0$, a two-pion sink at $t = \Delta$, and one of the operators Q_i^{lat} in the weak Hamiltonian inserted at all times $0 < t < \Delta$. We fit the correlation functions $C_{I,i}(\Delta, t)$,

$$C_{I,i}(\Delta, t) \approx M_i^{\Delta I, \text{lat}} N_{\pi\pi} N_K e^{-E(\pi\pi)_I \Delta} e^{-(m_K - E(\pi\pi)_I) t} \quad (2)$$

for $0 \ll t \ll \Delta$, using a one parameter exponential fit to determine the matrix elements $M_i^{\Delta I, \text{lat}}$. All these correlation functions can be expressed in terms of the 48 contractions enumerated in Section IV of [3] and labelled ① through ④⑧. The contractions are functions of Δ and t , but we leave this dependence implicit, writing for example $C_{2,1}(\Delta, t) = i\sqrt{2/3}\{\textcircled{1} + \textcircled{2}\}$.

The renormalization factors $Z_{ij}^{\text{lat} \rightarrow \overline{\text{MS}}}$ provide the connection between the bare lattice operators and those renormalized in the $\overline{\text{MS}}$ -NDR scheme at the scale μ ,

$$Q_i^{\overline{\text{MS}}}(\mu) = Z_{ij}^{\text{lat} \rightarrow \overline{\text{MS}}}(\mu, a) Q_j^{\text{lat}}(a). \quad (3)$$

The operators Q_i on the left of (3) correspond to the conventional 10-operator “physical” basis, which is over-complete (see e.g. [12]). When calculating the renormalization factors, it is convenient to work in an equivalent “chiral” basis of 7 linearly independent operators Q_j' with definite $SU(3)_L \times SU(3)_R$ transformation properties (see eqs.(172)-(175) in [12]). $z_i(\mu) + \tau y_i(\mu)$ are Wilson coefficient functions. F_I is the Lellouch-Lüscher factor relating the finite-volume Euclidean-space matrix element to the physical decay amplitude [13].

Evaluation of $\text{Re}A_2$: A_2 receives contributions from the Electroweak Penguin (EWP) operators Q_7 and Q_8 as well as a single operator $Q_{(27,1)}^{3/2}$,

$$Q_{(27,1)}^{3/2} = (\bar{s}^i d^i)_L \{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L, \quad (4)$$

where the superscript 3/2 denotes ΔI and the subscript (27, 1) denotes how the operator transforms under $SU(3)_L \times SU(3)_R$ chiral symmetry. i, j are color labels and the spinor indices are contracted within each pair of parentheses. The subscript L denotes *left*, so that e.g. $(\bar{s}^i d^i)_L (\bar{u}^j u^j)_L = (\bar{s}^i \gamma^\mu (1 - \gamma^5) d^i) (\bar{u}^j \gamma_\mu (1 - \gamma^5) u^j)$. The $\Delta I = 3/2$ components of the operators Q_1, Q_2, Q_9 and Q_{10} are all proportional to $Q_{(27,1)}^{3/2}$. From all our simulations we confirm that the contribution from the EWP operators to $\text{Re}A_2$ is about 1%; e.g. for physical kinematics we find $\text{Re}A_2 = (1.381 \pm 0.046 \pm 0.258) 10^{-8}$ GeV to which the EWP operators contribute $-0.0171 10^{-8}$ GeV [4, 5] (the physical value is $\text{Re}A_2 = 1.479(4) 10^{-8}$ GeV). We therefore neglect the EWP operators in the following discussion. Chiral symmetry implies that $Q_{(27,1)}^{3/2}$ does not mix with the EWP operators so that $\text{Re}A_2$ is proportional to its lattice matrix element (the constant of proportionality is the product of the Wilson coefficient, the

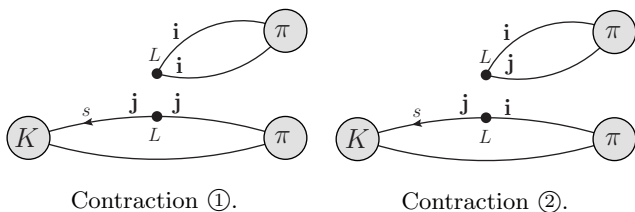


FIG. 1: The two contractions contributing to $\text{Re}A_2$. They are distinguished by the color summation (i, j denote color). s denotes the strange quark and L that the currents are left-handed.

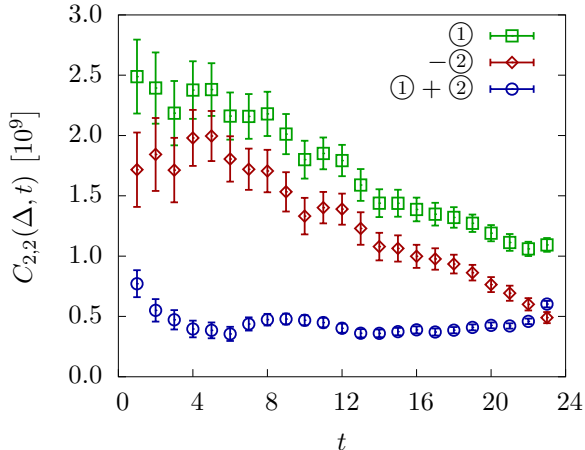


FIG. 2: Contractions ①, $-\text{②}$ and $\text{①} + \text{②}$ as functions of t from the simulation at physical kinematics and with $\Delta = 24$.

renormalization constant, finite-volume effects and kinematical factors; see [5] for a detailed discussion).

Fierz transformations allow the $K \rightarrow \pi\pi$ correlation function of $Q_{(27,1)}^{3/2}$ to be reduced to the sum of the two contractions illustrated in Fig. 1, labeled by ① and ②. The two contractions are identical except for the way that the color indices are summed. A_2 is proportional to the matrix element extracted from the sum $\text{①} + \text{②}$. The main message of this letter is our observation from all three simulations that ① and ② have opposite signs and are comparable in size. This is illustrated in Fig. 2 for the results at physical kinematics from [4, 5], where we plot ①, $-\text{②}$ and $\text{①} + \text{②}$ as functions of t . We extract A_2 by fitting $\text{①} + \text{②}$ in the interval $t \in [5, 19]$ where there is a significant cancellation between the two terms. A similar, although not quite so pronounced cancellation occurs at threshold for physical masses and for the heavier masses studied in [3, 10], see Fig. 3 for example.

It has been argued that the factorisation hypothesis [14] works reasonably well in reproducing the experimental value of A_2 (see e.g. Sec. VIII-4 in [15]). In this approach, the gluonic interactions between the quarks

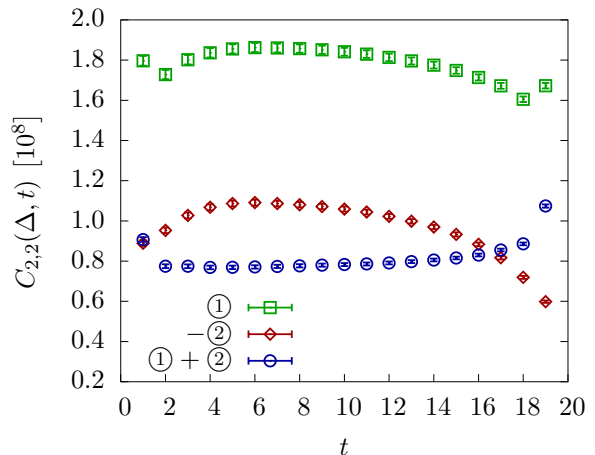


FIG. 3: Contractions ①, $-\text{②}$ and $\text{①} + \text{②}$ as functions of t from the simulation at threshold with $m_\pi \simeq 330$ MeV and $\Delta = 20$.

combining into different pions are neglected and A_2 is related to the decay constant f_π and the $K_{\ell 3}$ form factor close to zero momentum transfer. On the basis of color counting, one might therefore expect that $\text{②} \simeq 1/3 \text{①}$, whereas, for physical kinematics, we find $\text{②} \simeq -0.7 \text{①}$ and that nevertheless $\text{①} + \text{②}$ leads to the correct result for A_2 . Thus the expectation based on the factorisation hypothesis proves to be unreliable here.

Following the discovery that ① and ② have opposite signs we examined separately the two contributions to the matrix element $\langle \bar{K}^0 | (\bar{s}d)_L (\bar{s}d)_L | K^0 \rangle$ which contains the non-perturbative QCD effects in neutral kaon mixing [11]. The two contributions correspond to Wick contractions in which the two quark fields in the K^0 interpolating operator are contracted i) with fields from the same current in $(\bar{s}d)_L (\bar{s}d)_L$ and ii) with one field from each of the two currents. Color counting and the vacuum insertion hypothesis suggest that the two contributions come in the ratio 1:1/3, whereas we find that in QCD they have the opposite sign.

We stress that it is only the correlation function $\text{①} + \text{②}$ which has a time behaviour corresponding to $E_{(\pi\pi)_2}$. Because the calculation is performed in a finite-volume $E_{(\pi\pi)_2} \neq E_{(\pi\pi)_0}$ and ① and ② individually have an isospin 0 component. If $E_{(\pi\pi)_2} = m_K$ then $\text{①} + \text{②}$ is independent of t away from the kaon and two-pion sources, and this is what we observe, particularly in Fig. 2 where the energies are matched most precisely.

We postpone a discussion of the implications of these results to the $\Delta I = 1/2$ rule until the next section, but we believe that the partial cancellation observed in the evaluation of A_2 is a significant component.

Evaluation of $\text{Re}A_0$: The evaluation of A_0 at physical kinematics has not yet been completed. The results presented here are obtained at threshold, with the two pi-

ons in their zero-momentum ground state with each pion at rest up to finite-volume effects. Even at threshold we have had to overcome many theoretical and technical problems, including the evaluation of the 48 contractions contributing to the correlation functions, the renormalization of the operators in the effective Hamiltonian, the subtraction of power divergences and the evaluation of the finite-volume corrections. The threshold calculations do not require however, the isolation of an excited state. The pions in a physical decay each have a non-zero momentum in the center-of-mass frame, which corresponds to an excited state in lattice calculations. Given the poor statistical signals after the subtraction of power divergences and the evaluation of disconnected diagrams, the evaluation of A_0 at physical kinematics is currently impracticable with standard techniques and is the main motivation for our development of G-parity boundary conditions [6–9].

With the two pions at threshold we find [3, 10]

$$\frac{\text{Re}A_0}{\text{Re}A_2} = \begin{cases} 9.1(2.1) & \text{for } m_K = 878 \text{ MeV}, m_\pi = 422 \text{ MeV} \\ 12.0(1.7) & \text{for } m_K = 662 \text{ MeV}, m_\pi = 329 \text{ MeV}. \end{cases} \quad (5)$$

While these results differ significantly from the observed value of 22.5, because the calculations are not performed at physical kinematics, there is nevertheless already a significant enhancement in the ratio and it is interesting to understand its origin. In Tab. II we present the contributions to $\text{Re}A_0$ from each of the lattice operators in the 24^3 simulation with $a^{-1} = 1.73(3) \text{ GeV}$ and from each $\overline{\text{MS}}$ -NDR operator at renormalization scale 2.15 GeV. In both cases, the dominant contribution comes from the current-current operators Q_2 .

Since in a finite-volume $E_{(\pi\pi)_2} \neq E_{(\pi\pi)_0}$, one cannot satisfy the condition $m_K = E_{\pi\pi}$ for both isospin channels simultaneously with the same quark masses. Here we quote results using the fixed meson masses quoted in Eq. (5), which is sufficient for our current discussion. For these masses $E_{(\pi\pi)_0} = 766(29) \text{ MeV}$ ($629(15) \text{ MeV}$), $E_{(\pi\pi)_2} = 876(15) \text{ MeV}$ ($668(11) \text{ MeV}$) for the 16^3 (24^3) lattice. A study that interpolates in the kaon mass to make both decays energy-conserving may be found in [3].

The dominant contribution from the lattice operator Q_2 to the $\Delta I = 1/2$ correlation function is proportional to the contractions $2 \cdot \textcircled{1} - \textcircled{2}$ and corresponds to *type1* diagrams in the language of [3] (see Fig. 3 in [3]). In Fig. 4 we show the total contribution of Q_2 to the correlation function, as well as the total connected contribution and that of *type1* diagrams given by $\frac{i}{\sqrt{3}}\{2 \cdot \textcircled{1} - \textcircled{2}\}$. The errors on the total contribution are dominated by the disconnected diagrams. The observation that $\textcircled{1}$ and $\textcircled{2}$ have opposite signs leads to an enhancement between the two terms rather than the suppression in the factorization approximation $\textcircled{2} = \frac{1}{3}\textcircled{1}$. Similarly, in the case of Q_1 , the *type1* combination $\frac{i}{\sqrt{3}}\{2 \cdot \textcircled{2} - \textcircled{1}\}$ is dominant. In

i	Q_i^{lat} [GeV]	$Q_i^{\overline{\text{MS-NDR}}}$ [GeV]
1	$8.1(4.6) \cdot 10^{-8}$	$6.6(3.1) \cdot 10^{-8}$
2	$2.5(0.6) \cdot 10^{-7}$	$2.6(0.5) \cdot 10^{-7}$
3	$-0.6(1.0) \cdot 10^{-8}$	$5.4(6.7) \cdot 10^{-10}$
4	–	$2.3(2.1) \cdot 10^{-9}$
5	$-1.2(0.5) \cdot 10^{-9}$	$4.0(2.6) \cdot 10^{-10}$
6	$4.7(1.7) \cdot 10^{-9}$	$-7.0(2.4) \cdot 10^{-9}$
7	$1.5(0.1) \cdot 10^{-10}$	$6.3(0.5) \cdot 10^{-11}$
8	$-4.7(0.2) \cdot 10^{-10}$	$-3.9(0.1) \cdot 10^{-10}$
9	–	$2.0(0.6) \cdot 10^{-14}$
10	–	$1.6(0.5) \cdot 10^{-11}$
$\text{Re}A_0$	$3.2(0.5) \cdot 10^{-7}$	$3.2(0.5) \cdot 10^{-7}$

TABLE II: Contributions from each operator to $\text{Re}A_0$ for $m_K = 662 \text{ MeV}$ and $m_\pi = 329 \text{ MeV}$. The second column contains the contributions from the 7 linearly independent lattice operators with $1/a = 1.73(3) \text{ GeV}$ and the third column those in the 10-operator basis in the $\overline{\text{MS}}$ -NDR scheme at $\mu = 2.15 \text{ GeV}$. Numbers in parentheses represent the statistical errors.

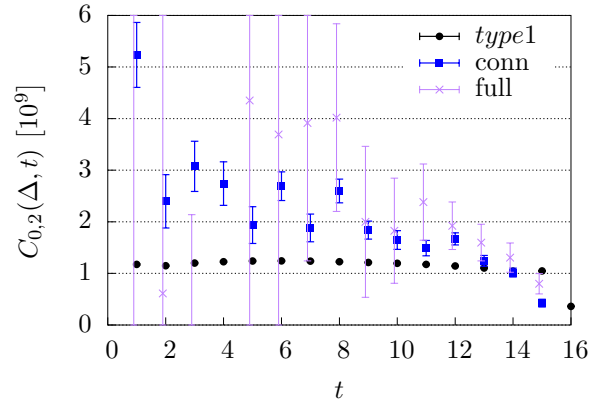


FIG. 4: Contributions of Q_2^{lat} to $\text{Re}A_0$ (purple crosses). The blue squares and black circles denote the connected and *type1* contractions respectively.

this case both the correlation function and the Wilson coefficient $z_1(\mu) + \tau y_1(\mu)$ are negative, so that the overall contribution adds to that from the correlation function of Q_2 .

Finally we note that in our data $\text{Re}A_2$ shows a much stronger mass dependence than $\text{Re}A_0$, which was also expected in $\text{SU}(2)$ chiral perturbation theory [16]. We attribute this to the partial cancellation between $\textcircled{1}$ and $\textcircled{2}$ in $\text{Re}A_2$. Our results for $\text{Re}A_2$ and $\text{Re}A_0$ are given in Tab. I.

Conclusions

From our recent computations of $K \rightarrow \pi\pi$ decay amplitudes a likely explanation of the $\Delta I = 1/2$ rule is emerging. In particular, we find that in the evaluation of

$\text{Re}A_2$, which is proportional to the sum of two contractions ① + ②, there is a significant cancellation between the two terms. The naïve expectation based on the factorization hypothesis suggests that ② $\approx \frac{1}{3}$ ①, whereas in QCD we find that they have the opposite sign. (The two terms contributing to B_K similarly have opposite signs contradicting expectations from the vacuum insertion approximation.)

The evaluation of A_0 at physical kinematics has not yet been performed. Our simulations at threshold with $m_\pi = 329 \text{ MeV}$ and 422 MeV show that the dominant contributions to A_0 comes from the current-current operators, with only small corrections from the penguin operators. This is true whether we express the results in terms of the bare lattice operators at $a^{-1} = 1.73 \text{ GeV}$ or the $\overline{\text{MS}}$ -NDR renormalized operators at $\mu = 2.15 \text{ GeV}$ (see Tab. II). Although 48 contractions contribute to the $I = 0$ correlation function, in our simulations the largest contributions again come from contractions ① and ② with relative signs which enhance $\text{Re}A_0$.

References to estimates of the amplitudes using analytic or model approximations are presented in the reviews [17, 18]. We note that a suppression of $\text{Re}A_2$ and an enhancement of $\text{Re}A_0$ was found in [19] using the $1/N$ expansion with a particular ansatz for matching the short and long-distance factors at scales $0.6\text{--}0.8 \text{ GeV}$.

The results presented above indicate that $\text{Re}A_2$ is very sensitive to the choice of quark masses and momenta; a sensitivity we attribute to the partial cancellation of the two contributing contractions. On the other hand, there is no such cancellation in $\text{Re}A_0$ and indeed the results depend much less on the masses and the values we find are already close to the experimental result. Of course before we can claim to understand the $\Delta I = 1/2$ rule quantitatively, we need to reproduce $\text{Re}A_0/\text{Re}A_2=22.5$ at physical quark masses and kinematics and we are currently undertaking this challenge. Nevertheless, from the results and discussion of this paper it appears that, in addition to the well known perturbative enhancement of $\text{Re}A_0/\text{Re}A_2$, the explanation is a combination of a significant relative suppression of $\text{Re}A_2$ as well as some enhancement of $\text{Re}A_0$ with penguin operators contributing very little.

Acknowledgements We thank W. Bardeen and A. Buras for informative discussions. P. Boyle was supported in part by STFC grants ST/J000329/1,

ST/K005804/1, ST/K000411/1 and ST/H008845/1, N. Christ, Q. Liu and D. Zhang by U.S. DOE grant DE-FG02-92ER40699, E. Goode, T. Janowski, A. Lytle and C. Sachrajda by STFC Grant ST/G000557/1, C. Lehner by the RIKEN FPR program and A. Soni by US DOE grant DE-AC02-98CH10886(BNL).

-
- [1] M. Gaillard and B. W. Lee, Phys.Rev.Lett., **33**, 108 (1974).
 - [2] G. Altarelli and L. Maiani, Phys.Lett., **B52**, 351 (1974).
 - [3] T. Blum, P. Boyle, N. Christ, N. Garron, E. Goode, *et al.*, Phys.Rev., **D84**, 114503 (2011), arXiv:1106.2714 [hep-lat].
 - [4] T. Blum, P. Boyle, N. Christ, N. Garron, E. Goode, *et al.*, Phys.Rev.Lett., **108**, 141601 (2012), arXiv:1111.1699 [hep-lat].
 - [5] T. Blum, P. Boyle, N. Christ, N. Garron, E. Goode, *et al.*, Phys.Rev., **D86**, 074513 (2012), arXiv:1206.5142 [hep-lat].
 - [6] U. Wiese, Nucl.Phys., **B375**, 45 (1992).
 - [7] C.-h. Kim and N. H. Christ, Nucl.Phys.Proc.Suppl., **119**, 365 (2003), arXiv:hep-lat/0210003 [hep-lat].
 - [8] C. Kim, Nucl.Phys.Proc.Suppl., **129**, 197 (2004), arXiv:hep-lat/0311003 [hep-lat].
 - [9] C. Kim and N. H. Christ, PoS, **LAT2009**, 255 (2009), arXiv:0912.2936 [hep-lat].
 - [10] Q. Liu, *Kaon to Two Pion decays from Lattice QCD: $\Delta I = 1/2$ rule and CP violation*, Ph.D. thesis (2012), Columbia University.
 - [11] R. Arthur, T. Blum, P. A. Boyle, N. H. Christ, N. Garron, *et al.* (RBC-UKQCD Collaboration), (2012), arXiv:1208.4412 [hep-lat].
 - [12] T. Blum *et al.* (RBC Collaboration), Phys.Rev., **D68**, 114506 (2003), arXiv:hep-lat/0110075 [hep-lat].
 - [13] L. Lellouch and M. Lüscher, Commun.Math.Phys., **219**, 31 (2001).
 - [14] M. Gaillard and B. W. Lee, Phys.Rev., **D10**, 897 (1974).
 - [15] J. Donoghue, E. Golowich, and B. R. Holstein, Camb.Monogr.Part.Phys.Nucl.Phys.Cosmol., **2**, 1 (1992).
 - [16] J. Bijnens and A. Celis, Phys.Lett., **B680**, 466 (2009), arXiv:0906.0302 [hep-ph].
 - [17] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, Rev.Mod.Phys., **68**, 1125 (1996), arXiv:hep-ph/9512380 [hep-ph].
 - [18] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich, and J. Portoles, Rev.Mod.Phys., **84**, 399 (2012), arXiv:1107.6001 [hep-ph].
 - [19] W. A. Bardeen, A. Buras, and J. Gerard, Phys.Lett., **B192**, 138 (1987).